MONOTONICITY OF LYAPUNOV TYPE FUNCTIONS FOR MEASURE–DRIVEN DIFFERENTIAL SYSTEMS

Olga Samsonyuk
Matrosov Institute for System Dynamics and
Control Theory of Siberian Branch of
the Russian Academy of Sciences
Russia
samsonyuk.olga@gmail.com

Abstract
This paper deals with impulsive dynamical systems with trajectories of bounded variation and impulsive controls (regular vector measures). Definitions of strong and weak monotonicity and V-monotonicity of Lyapunov type functions with the corresponding infinitesimal conditions in the form of Hamilton–Jacobi inequalities are proposed.

Key words
Measure-driven impulsive control system, trajectories of bounded variation, monotonicity of Lyapunov type functions.

1 Introduction

This paper deals with the nonlinear impulsive control systems that formally can be described as follows

$$dx(t) = f\big(t, x(t), u(t)\big)dt + G\big(t, x(t)\big)\pi(\mu), \quad (1)$$

$$u(t) \in U \text{ a.e. on } T, \quad \pi(\mu) \in \mathcal{W}(T, K). \quad (2)$$

Here, T is a time interval in R, U is a compact set in R^r, K is a convex closed cone in R^m, $x(\cdot) \in BV(T, R^n)$, where $BV\big(T, R^n\big)$ is the Banach space of R^n-valued functions of bounded variation on T, $u(\cdot)$ is an ordinary control, and $\pi(\mu) := \big(\mu, \gamma(\mu)\big)$ is an impulsive control such that

i) μ is a K-valued bounded Borel measure on T;

ii) $\gamma(\mu)$ is the set $\{d_s, \omega_s(\cdot)\}_{s\in S}$, where components satisfy the following conditions:

(a) $S \supseteq S_{\mathrm{d}}(\mu) := \{s \in T \mid \mu(\{s\}) \neq 0\}$, S is at most denumerable set in T;

(b) $\forall\, s \in S \;\; d_s \geq 0$, $\omega_s : [0, d_s] \to co\; K_1$,

$$d_s \geq ||\mu\big(\{s\}\big)||, \;\; \int_0^{d_s} \omega_s(\tau)d\tau = \mu\big(\{s\}\big);$$

(c) $\displaystyle\sum_{s\in S} d_s < \infty$.

Here, $K_1 = \{v \in K \mid ||v|| = 1\}$, $||v|| = \sum_{j=1}^m |v_j|$, and $co\, A$ is the convex hull of a set A.

The set of the impulsive controls, $\mathcal{W}(T, K)$, consists of all $\pi(\mu)$ satisfying conditions i), ii). The solution concept of (1), (2) was stated in [Samsonyuk, 2015; Samsonyuk, 2014] and is a modification of the notion of generalized solution introduced in [Miller and Rubinovich, 2003; Miller, 1993; Miller, 1996] and, for $K = R^m$, [Zavalishchin and Sesekin, 1991; Zavalishchin and Sesekin, 1997]. Also it closely concerns the solution concept from [Motta and Rampazzo, 1995; Motta and Rampazzo, 1996] and [Arutyunov, Karamzin, and Pereira, 2010; Bressan and Rampazzo, 1988; Pereira and Silva, 2000]. This concept is presented in detail in the next section.

In this paper, we focus mainly on the study of some monotonicity properties of a continuous Lyapunov type function relative to the impulsive control system (1), (2). Definitions of strong and weak monotonicity and V-monotonicity are proposed and discussed. The set of conventional variables t and x of Lyapunov type functions is now supplemented with a variable V, which, on the one hand, is responsible for the impulsive dynamics of (1), (2) and has the property of a time variable and, on the other hand, characterizes some resource of the impulsive control. We will show that such double interpretation of variable V leads to different definitions of monotonicity, which are called monotonicity and V-monotonicity. For Lyapunov type functions, infinitesimal conditions of monotonicity in the form of Hamilton–Jacobi type proximal or differential inequalities are presented.

This paper is organized as follows. The solution concept of (1), (2) is given in Section 2. In Section 3, definitions of strong and weak monotonicity and V-monotonicity of Lyapunov type functions are proposed and discussed. Sections 4 and 5 are devoted to the infinitesimal conditions of monotonicity in the form of Hamilton–Jacobi proximal and differential inequalities.

2 The solution concept

In this section, we describe the set of solution of (1), (2) as a closure of the set of absolutely continuous solutions, where the closure is understood in the sense of the convergence in Hausdorff metric for graphs of a supplemented absolutely continuous trajectories.

Suppose that the following assumptions hold.

(H1) The functions $f(t,x,u)$, $G(t,x)$ are continuous; for any compact set $Q \subset \mathbb{R}^n$ there exist constants L_1, $L_2 > 0$ such that

$$|f(t,x_1,u) - f(t,x_2,u)| \le L_1|x_1 - x_2|,$$
$$|G(t,x_1) - G(t,x_2)| \le L_2|x_1 - x_2|$$

whenever $(t,x_1,u),(t,x_2,u) \in T \times Q \times U$; moreover, there exist constants $c_1,\ c_2 > 0$ such that

$$|f(t,x,u)| \le c_1(1+|x|),\ \ |G(t,x)| \le c_2(1+|x|)$$

whenever $(t,x,u) \in T \times \mathbb{R}^n \times U$. Here, $|\cdot|$ denotes a vector or consistent matrix norm.

(H2) The set $f(t,x,U)$ is a convex set $\forall\ (t,x) \in T \times \mathbb{R}^n$.

Let us consider a conventional kind system governed by the standard control dynamics on $T = [t_0, t_1]$

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) + G\big(t, x(t)\big)v(t), \qquad (3)$$

$$u(t) \in U, \quad v(t) \in K \quad \text{a.e. on } T. \qquad (4)$$

Here, $x(\cdot)$ is an absolutely continuous function such that $x(t_0) = x_0 \in \mathbb{R}^n$, $u(\cdot)$ and $v(\cdot)$ are $\mathcal{L}$-measurable bounded vector-functions.

Let $V(\cdot)$ be defined by the rule

$$V(t) = \int_t^{t_1} ||v(\xi)|| d\xi. \qquad (5)$$

A pair of functions $\big(x(\cdot), V(\cdot)\big) =: \varkappa_V^{\mathrm{ac}}(\cdot)$ is called a supplemented trajectory of (3), (4) if $x(\cdot)$ is a solution of (3), (4) for some $u(\cdot)$ and $v(\cdot)$ and $V(\cdot)$ is defined by (5).

Let A and B be non-empty compact subsets of R^{n+2}. Define Hausdorff distance $d(A,B)$ between A and B by the rule

$$d(A,B) = \min\{\varepsilon \ge 0\ :\ A \subset B^\varepsilon,\ \ B \subset A^\varepsilon\},$$

where

$$A^\varepsilon := \bigcup_{x \in A} B_\varepsilon(x), \quad B^\varepsilon := \bigcup_{x \in B} B_\varepsilon(x),$$

and $B_\varepsilon(x)$ is the closed ball of radius ε and center x.

Define the following objects:

$\operatorname*{graph}_T \varkappa_V^{ac} := \Big\{(t,x,V)\ :\ t \in T,\ (x,V) = \varkappa_V^{ac}(t)\Big\}$;

$\mathcal{G}_T^{ac}(t_0,x_0) := \Big\{\operatorname*{graph}_T \varkappa_V^{ac} \mid \varkappa_V^{ac}(\cdot) \text{ satisfies (3)–(5)}\Big\}$.

Let $\mathcal{G}_T(t_0,x_0)$ be the closure of set $\mathcal{G}_T^{ac}(t_0,x_0)$ in sense of Hausdorff metric. Now let us consider set-valued functions $\varkappa_V : T \hookrightarrow \mathbb{R}^n \times \mathbb{R}_+$ such that their graphs are in $\mathcal{G}_T(t_0,x_0)$. Denote by $\mathcal{T}(t_0,x_0)$ the set of all such set-valued functions, i.e.

$\mathcal{T}(t_0,x_0) := \Big\{\varkappa_V : T \hookrightarrow \mathbb{R}^n \times \mathbb{R}_+ \mid \operatorname*{graph}_T \varkappa_V \in \mathcal{G}_T(t_0,x_0)\Big\}$.

Definition 1. Let a set-valued function $\varkappa_V$ be in $\mathcal{T}(t_0,x_0)$. Then $\varkappa_V$ is said to be a generalized solution of (3), (4) as well as a supplemented trajectory of impulsive control system (1), (2).

Note that any selection $\big(x(\cdot),V(\cdot)\big)$ of $\varkappa_V$ is a function of bounded variation on $T = [t_0,t_1]$.

Let μ be a K-valued bounded Borel measure on T. Given μ, let $S_{\mathrm{d}}(\mu)$ be the set $\{s \in T \mid \mu(\{s\}) \ne 0\}$, μ_c be the continuous component in the Lebesgue decomposition of μ, and $|\mu_{\mathrm{c}}|$ be a total variation of the measure μ_{c}. In [Samsonyuk, 2014] the following result was proved.

Lemma 1. 1) *Let $\varkappa_V$ be in $\mathcal{T}(t_0,x_0)$. Then there exists $u(\cdot)$, $\pi(\mu)$ satisfying* (2) *such that the following conditions hold:*

i) $\forall\, t \in T/S \qquad \varkappa_V(t) = \big\{\big(\tilde{x}(t), \tilde{V}(t)\big)\big\}$, where $\tilde{x}(\cdot)$ and $\tilde{V}(\cdot)$ are defined by the rule:

$$\begin{aligned}
&\tilde{x}(t_0) = x_0,\\
&\tilde{x}(t) = x_0 + \int_{t_0}^{t} f\big(t,\tilde{x}(t),u(t)\big)dt + \\
&\quad + \int_{t_0}^{t} G\big(t,\tilde{x}(t)\big)\mu_{\mathrm{c}}(dt)\\
&\quad + \sum_{s \le t,\ s \in S} \Big(\tilde{x}(s) - \tilde{x}(s-)\Big), \quad t \in (t_0,t_1],
\end{aligned} \qquad (6)$$

where, for each $s \in S$, $\tilde{x}(s) = z_s(d_s)$ and $z_s(\cdot)$ is a solution of the differential equation

$$\begin{aligned}
\frac{dz_s(\tau)}{d\tau} &= G\big(s, z_s(\tau)\big)\omega_s(\tau),\\
z_s(0) &= \tilde{x}(s-), \quad \tau \in [0,d_s],
\end{aligned} \qquad (7)$$

$$\begin{aligned}
\tilde{V}(t) = |\mu_{\mathrm{c}}([t,t_1])| + \sum_{s \ge t,\ s \in S} d_s,\\
t \in [t_0,t_1), \quad \tilde{V}(t_1) = 0;
\end{aligned} \qquad (8)$$

ii) $\forall\, s \in S$

$$\varkappa_V(s) = \big\{\big(z_s(\tau)), \tilde{V}(s) - \tau\big) \mid \tau \in [0, d_s]\big\}. \quad (9)$$

2) *Let a set-valued function* $\varkappa_V : [t_0, t_1] \hookrightarrow \mathbb{R}^{n+1}$ *satisfy conditions i), ii) for some* $u(\cdot), \pi(\mu)$*. Then* $\varkappa_V \in \mathcal{T}(t_0, x_0)$*.*

In [Samsonyuk, 2015] the following result was proved.

Lemma 2. *A set-valued function* $\varkappa_V : T \hookrightarrow \mathbb{R}^n \times \mathbb{R}_+$ *is a supplemented trajectory of* (1), (2) *iff for any selection* $\big(x(\cdot), V(\cdot)\big)$ *of* $\varkappa_V$ *there exists* $\big\{x_k(\cdot), V_k(\cdot), u_k(\cdot), v_k(\cdot)\big\}$ *such that*

i) functions $x_k(\cdot)$*,* $V_k(\cdot)$*,* $u_k(\cdot)$*, and* $v_k(\cdot)$ *satisfy* (3), (4)*;*

ii) $\sup_k ||v_k(\cdot)||_{L_1} < \infty$*;*

iii) $x_k(t) \to x(t),\ V_k(t) \to V(t)\ \forall\, t \in [t_0, t_1]$*.*

Note that the generalized solutions of (3), (4) (and consequently supplemented trajectories of (1), (2)) can be considered in the reverse time; i. e. for $t \le t_1$. In this paper we will consider (1), (2) both forward and backward in time. Then for the system considered forward in time a supplemented trajectory will be denoted by $\varkappa_V^+$ and named right supplemented trajectory; for the system considered backward in time a supplemented trajectory will be denoted by $\varkappa_V^-$ and named a left supplemented trajectory.

Now let us summarize the solution concept by the following definition.

Definition 2. Let $u(\cdot)$ and $\pi(\mu) = (\mu, \gamma(\mu))$ satisfy (2) for $T = [t_0, t_1]$. The set-valued functions $\varkappa_V^- : [t_0, t_1] \hookrightarrow \mathbb{R}^{n+1}$, $\varkappa_V^+ : [t_0, t_1] \hookrightarrow \mathbb{R}^{n+1}$ are said to be left and right supplemented trajectories of (1), (2) corresponding to $u(\cdot)$, $\pi(\mu)$, and an initial condition $x_0 \in \mathbb{R}^n$ if the following conditions hold:

i) $\forall\, t \in T/S$

$$\varkappa_V^-(t) = \big\{\big(\tilde{x}(t), V^-(t)\big)\big\},$$
$$\varkappa_V^+(t) = \big\{\big(\tilde{x}(t), V^+(t)\big)\big\},$$

where $\tilde{x}(\cdot)$ is defined by (6),

$$V^-(t) = |\mu_c([t_0, t])| + \sum_{s \le t,\, s \in S} d_s,$$
$$t \in (t_0, t_1], \quad V^-(t_0) = 0,$$
$$V^+(t) = |\mu_c([t, t_1])| + \sum_{s \ge t,\, s \in S} d_s,$$
$$t \in [t_0, t_1), \quad V^+(t_1) = 0;$$

ii) $\forall\, s \in S$

$$\varkappa_V^-(s) = \big\{\big(z_s(\tau; \tilde{x}(s-)), V^-(s-) + \tau\big) \mid \tau \in [0, d_s]\big\},$$
$$\varkappa_V^+(s) = \big\{\big(z_s(\tau; \tilde{x}(s-)), V^+(s) - \tau\big) \mid \tau \in [0, d_s]\big\}.$$

Let

$$\varkappa_V^-(t+) := \big\{\big(\tilde{x}(t), V^-(t)\big)\big\},\ \ t \in (t_0, t_1],$$
$$\varkappa_V^+(t-) := \big\{\big(\tilde{x}(t-), V^+(t)\big)\big\}\ \forall\, t \in [t_0, t_1).$$

3 Definitions of monotonicity

In this section, definitions of strong and weak monotonicity and V-monotonicity of Lyapunov type functions relative to the impulsive system (1), (2) are proposed.

Let $(t, x, V) \to \varphi(t, x, V)$ be of a continuous function. Let us begin with definition of strong and weak V-monotonicity.

Let $(t_\alpha, x_\alpha) \in [a, b] \times \mathbb{R}^n$, $V_\alpha \ge 0$. Define two sets of supplemented trajectories:

$$\mathcal{T}^-_{V_\alpha}(t_\alpha, x_\alpha) =$$
$$= \big\{\varkappa_V^-,\ \ t \in [a, t_\alpha] \ \big|\ \varkappa_V^-(t_\alpha+) = \{(x_\alpha, V_\alpha)\}\big\},$$

$$\mathcal{T}^+_{V_\alpha}(t_\alpha, x_\alpha) =$$
$$= \big\{\varkappa_V^+,\ \ t \in [t_\alpha, b] \ \big|\ \varkappa_V^+(t_\alpha-) = \{(x_\alpha, V_\alpha)\}\big\}.$$

Thereby set $\mathcal{T}^-_{V_\alpha}(t_\alpha, x_\alpha)$ consists of the supplemented trajectories, $\varkappa_V^-$, considered backward in time $t \le t_\alpha$ and started from x_α at the time t_α and set $\mathcal{T}^+_{V_\alpha}(t_\alpha, x_\alpha)$ consists of the supplemented trajectories, $\varkappa_V^+$, considered forward in time $t \ge t_\alpha$ and started from x_α at the time t_α. In other words, for given $(t_\alpha, x_\alpha, V_\alpha)$, the graph of $\varkappa_V \in \mathcal{T}^-_{V_\alpha}(t_\alpha, x_\alpha)$ finishes at given point $(t_\alpha, x_\alpha, V_\alpha)$ but the graph of $\varkappa_V \in \mathcal{T}^+_{V_\alpha}(t_\alpha, x_\alpha)$ starts at this point. On the corresponding interval $[a, t_\alpha]$ or $[t_\alpha, b]$ trajectory $\varkappa_V$ is approximated by a sequence of absolutely continuous trajectories of (3), (4) with the resource of impulsive control V_α.

Let $M \ge 0$. Below, M is permitted to be $M = +\infty$; in this case, $[0, M] := [0, +\infty)$.

Let $\underset{[t_0,t_1]}{\operatorname{graph}} \varkappa_V$ be the graph of a set-valued function $\varkappa_V$ on interval $[t_0, t_1] \subseteq [a, b]$; i.e. $\underset{[t_0,t_1]}{\operatorname{graph}} \varkappa_V =$

$$\{(t, x, V) \mid t \in [t_0, t_1],\ (x, V) = \varkappa_V(t)\}.$$

Let $Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$ consist of $(t, x, V) \in [a, b] \times \mathbb{R}^n \times [0, M]$ such that

$$\varphi(t, x, V) \ge \varphi(t_\alpha, x_\alpha, V_\alpha).$$

Definition 3. φ is *strongly V-increasing* if for any $(t_\alpha, x_\alpha) \in [a, b] \times \mathbb{R}^n$, $V_\alpha \in [0, M]$ and any $\varkappa_V \in \mathcal{T}^+_{V_\alpha}(t_\alpha, x_\alpha)$ the following inclusion

$$\underset{[t_\alpha,b]}{\operatorname{graph}} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha) \quad (10)$$

is fulfilled.

Definition 4. φ is *weakly V-increasing* if for any $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$ and $V_\alpha \in [0, M]$ there exists $\varkappa_V \in \mathcal{T}^+_{V_\alpha}(t_\alpha, x_\alpha)$ such that inclusion (10) is fulfilled.

The next two definitions give V-monotonicity conditions considered backward in time.

Definition 5. φ is *strongly V-predecreasing* if for any $(t_\alpha, x_\alpha) \in [a, b] \times \mathbb{R}^n$, $V_\alpha \in [0, M]$ and any $\varkappa_V \in \mathcal{T}^-_{V_\alpha}(t_\alpha, x_\alpha)$ the following inclusion

$$\operatorname*{graph}_{[a,t_\alpha]} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha) \tag{11}$$

is fulfilled.

Definition 6. φ is *weakly V-predecreasing* if for any $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$ and $V_\alpha \in [0, M]$ there exists $\varkappa_V \in \mathcal{T}^-_{V_\alpha}(t_\alpha, x_\alpha)$ such that inclusion (11) is fulfilled.

Let us briefly comment on these definitions. Let $Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$ be the set

$$Q^+_\varphi(t_\alpha, x_\alpha, V_\alpha) \bigcup Q^-_\varphi(t_\alpha, x_\alpha, V_\alpha),$$

where for elements of $Q^+_\varphi(t_\alpha, x_\alpha, V_\alpha)$ we have $t \geq t_\alpha$; for $Q^-_\varphi(t_\alpha, x_\alpha, V_\alpha)$, $t \leq t_\alpha$. Then φ has the property of strong or weak V-monotonicity iff set $Q^+_\varphi(t_\alpha, x_\alpha, V_\alpha)$ has the corresponding strong or weak V-invariance property relative to the impulsive system (1), (2). For the monotonicity property considered backward in time, it takes place for $Q^-_\varphi(t_\alpha, x_\alpha, V_\alpha)$ (see [Samsonyuk, 2015]).

Note that in Definition 3 and 4 we use right supplemented trajectories but in Definitions 5 and 6, left ones. It is not convenient in some case, for example, when φ is a value function in dynamic programming and needs both strong monotonicity forward in time and weak monotonicity backward in time.

Now let us define properties of strong and weak monotonicity. Denote by $\mathcal{T}^{[0,M]}_{[a,b]}$ and $\mathcal{T}^{[M,0]}_{[a,b]}$ correspondingly the sets of all left or right supplemented trajectories of (1), (2) defined on interval $[a, b]$ and such that its V-component is bounded by $M \geq 0$. We will consider the restriction of these sets for interval $[a, t_\alpha]$ or $[t_\alpha, b]$.

Definition 7. φ is *strongly increasing* if for any $(t_\alpha, x_\alpha) \in [a, b] \times \mathbb{R}^n$, $V_\alpha \in [0, M]$ and any $\varkappa_V \in \mathcal{T}^{[0,M]}_{[t_\alpha,b]}$ satisfying condition $\varkappa_V(t_\alpha-) = (x_\alpha, V_\alpha)$ the following inclusion

$$\operatorname*{graph}_{[t_\alpha,b]} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$$

is fulfilled.

Definition 8. φ is *weakly predecreasing* if for any $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$ and $V_\alpha \in [0, M]$ there exists $\varkappa_V \in \mathcal{T}^{[0,M]}_{[a,t_\alpha]}$ satisfying condition $\varkappa_V(t_\alpha+) = (x_\alpha, V_\alpha)$ and such that

$$\operatorname*{graph}_{[a,t_\alpha]} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha).$$

Definition 9. φ is *weakly increasing* if for any $(t_\alpha, x_\alpha) \in T \times \mathbb{R}^n$ and $V_\alpha \in [0, M]$ there exists $\varkappa_V \in \mathcal{T}^{[M,0]}_{[t_\alpha,b]}$ satisfying condition $\varkappa_V(t_\alpha-) = (x_\alpha, V_\alpha)$ and such that

$$\operatorname*{graph}_{[t_\alpha,b]} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha).$$

Definition 10. φ is *strongly predecreasing* if for any $(t_\alpha, x_\alpha) \in [a, b] \times \mathbb{R}^n$, $V_\alpha \in [0, M]$ and any $\varkappa_V \in \mathcal{T}^{[M,0]}_{[a,t_\alpha]}$ satisfying condition $\varkappa_V(t_\alpha+) = (x_\alpha, V_\alpha)$ the following inclusion

$$\operatorname*{graph}_{[a,t_\alpha]} \varkappa_V \subset Q_\varphi(t_\alpha, x_\alpha, V_\alpha)$$

is fulfilled.

Note that in Definitions 7-10 we say about the restriction of trajectories on $[t_\alpha, b]$ or $[a, t_\alpha]$.

It is easy to prove that if φ is strongly V-increasing then φ_1 defined by the rule

$$\varphi_1(t, x, V) = \varphi(t, x, M - V)$$

is strongly increasing. But if φ is weakly V-increasing then φ is also weakly increasing. For more details, see [Samsonyuk, 2014].

4 Monotonicity conditions for smooth Lyapunov type functions

Let us give brief summary of monotonicity conditions for smooth Lyapunov type functions. Let $\varphi(t, x, V)$ be a twice continuously differentiable function.

Define functions h_0, h_1, $\mathcal{H}_0$, and $\mathcal{H}_1$ by the rule: $h_0(t, x, \psi) = \min\limits_{u \in U} \langle \psi, f(t, x, u) \rangle$, $h_1(t, x, \psi) = \min\limits_{\omega \in K_1} \langle \psi, G(t, x)\omega \rangle$, $\mathcal{H}_0(t, x, \psi) = \max\limits_{u \in U} \langle \psi, f(t, x, u) \rangle$, $\mathcal{H}_1(t, x, \psi) = \max\limits_{\omega \in K_1} \langle \psi, G(t, x)\omega \rangle$. Note that these functions are counterparts of the lower and upper Hamiltonian with respect to ordinary and impulsive controls for (1), (2).

Define Hamilton–Jacobi type differential operators $\gamma[\varphi]$, $\Gamma[\varphi]$ $\gamma_-[\varphi]$, and $\Gamma_-[\varphi]$:

$$\begin{aligned}\gamma[\varphi](t, x, V) :=& \\ &\min_{\substack{\omega_0, \omega_1 \geq 0 \\ \omega_0 + \omega_1 = 1}} \big\{ \big(\varphi_t + h_0(t, x, \varphi_x)\big)\omega_0 + \\ &\qquad + \big(\varphi_V + h_1(t, x, \varphi_x)\big)\omega_1 \big\},\end{aligned}$$

$$\begin{aligned}\Gamma[\varphi](t, x, V) :=& \\ &\max_{\substack{\omega_0, \omega_1 \geq 0 \\ \omega_0 + \omega_1 = 1}} \big\{ \big(\varphi_t + \mathcal{H}_0(t, x, \varphi_x)\big)\omega_0 + \\ &\qquad + \big(\varphi_V + \mathcal{H}_1(t, x, \varphi_x)\big)\omega_1 \big\},\end{aligned}$$

$$\gamma_-[\varphi](t,x,V) := \min_{\substack{\omega_0,\omega_1\geq 0\\ \omega_0+\omega_1=1}} \big\{\big(\varphi_t + h_0(t,x,\varphi_x)\big)\omega_0 + \\ +\big(-\varphi_V + h_1(t,x,\varphi_x)\big)\omega_1\big\},$$

$$\Gamma_-[\varphi](t,x,V) := \max_{\substack{\omega_0,\omega_1\geq 0\\ \omega_0+\omega_1=1}} \big\{\big(\varphi_t + \mathcal{H}_0(t,x,\varphi_x)\big)\omega_0 + \\ +\big(-\varphi_V + \mathcal{H}_1(t,x,\varphi_x)\big)\omega_1\big\}.$$

We shall use the following notation: $\varphi^t(\cdot,\cdot) := \varphi(t,\cdot,\cdot)$, $\varphi^{V_0}(\cdot,\cdot) := \varphi(\cdot,\cdot,0)$. To simplify the record of statements, let the following agreements hold: the differential inequalities containing one of $\gamma[\varphi]$, $\Gamma[\varphi]$, $\gamma_-[\varphi]$, or $\Gamma_-[\varphi]$ are considered $\forall\ (t,x,V) \in (a,b) \times \mathbb{R}^n \times (0,M)$; the differential inequalities containing φ^{V_0} are considered $\forall\,(t,x) \in (a,b) \times \mathbb{R}^n$ and for φ^t, $\forall\,(x,V) \in \mathbb{R}^n \times (0,M)$ and given $t=a$ or $t=b$.

Monotonicity conditions for smooth Lyapunov type functions φ are given by the following differential inequalities.

Strong monotonicity conditions:

1) increasing or V-preincreasing $\Leftrightarrow$

$$\gamma[\varphi](t,x,V) \geq 0; \tag{12}$$

2) decreasing or V-predecreasing $\Leftrightarrow$

$$\Gamma[\varphi](t,x,V) \leq 0; \tag{13}$$

3) preincreasing or V-increasing $\Leftrightarrow$

$$\gamma_-[\varphi](t,x,V) \geq 0; \tag{14}$$

4) predecreasing or V-decreasing $\Leftrightarrow$

$$\Gamma_-[\varphi](t,x,V) \leq 0. \tag{15}$$

Weak monotonicity conditions:

5) predecreasing or V-predecreasing $\Leftrightarrow$

$$\begin{aligned} &\gamma[\varphi](t,x,V) \leq 0,\\ &\varphi_t^{V_0} + h_0(t,x,\varphi_x^{V_0}) \leq 0,\\ &\varphi_V^t + h_1(t,x,\varphi_x^t) \leq 0,\\ &\qquad\qquad\qquad t=a; \end{aligned} \tag{16}$$

6) preincreasing or V-preincreasing $\Leftrightarrow$

$$\begin{aligned} &\Gamma[\varphi](t,x,V) \geq 0,\\ &\varphi_t^{V_0} + \mathcal{H}_0(t,x,\varphi_x^{V_0}) \geq 0,\\ &\varphi_V^t + \mathcal{H}_1(t,x,\varphi_x^t) \geq 0,\\ &\qquad\qquad\qquad t=a; \end{aligned} \tag{17}$$

7) decreasing or V-decreasing $\Leftrightarrow$

$$\begin{aligned} &\gamma_-[\varphi](t,x,V) \leq 0,\\ &\varphi_t^{V_0} + h_0(t,x,\varphi_x^{V_0}) \leq 0,\\ &-\varphi_V^t + h_1(t,x,\varphi_x^t) \leq 0,\\ &\qquad\qquad\qquad t=b; \end{aligned} \tag{18}$$

8) increasing or V-increasing $\Leftrightarrow$

$$\begin{aligned} &\Gamma_-[\varphi](t,x,V) \geq 0,\\ &\varphi_t^{V_0} + \mathcal{H}_0(t,x,\varphi_x^{V_0}) \geq 0,\\ &-\varphi_V^t + \mathcal{H}_1(t,x,\varphi_x^t) \geq 0,\\ &\qquad\qquad\qquad t=b. \end{aligned} \tag{19}$$

Note that for strong monotonicity we have

$$\begin{aligned} \gamma[\varphi](t,x,V) \geq 0 \quad &\Leftrightarrow\\ &\varphi_t + h_0(t,x,\varphi_x) \geq 0,\\ &\varphi_V + h_1(t,x,\varphi_x) \geq 0; \end{aligned}$$

$$\begin{aligned} \Gamma[\varphi](t,x,V) \leq 0 \quad &\Leftrightarrow\\ &\varphi_t + \mathcal{H}_0(t,x,\varphi_x) \leq 0,\\ &\varphi_V + \mathcal{H}_1(t,x,\varphi_x) \leq 0. \end{aligned}$$

But, for weak monotonicity,

$$\begin{aligned} &\gamma[\varphi](t,x,V) \leq 0 \quad \Leftrightarrow\\ &\min\big\{\varphi_t + h_0(t,x,\varphi_x); \varphi_V + h_1(t,x,\varphi_x)\big\} \leq 0; \end{aligned}$$

$$\begin{aligned} &\Gamma[\varphi](t,x,V) \geq 0 \quad \Leftrightarrow\\ &\max\big\{\varphi_t + \mathcal{H}_0(t,x,\varphi_x); \varphi_V + \mathcal{H}_1(t,x,\varphi_x)\big\} \geq 0. \end{aligned}$$

Thereby the set-valued maps $(t,x,V) \to \omega_0(t,x,V)$, and $(t,x,V) \to \omega_1(t,x,V)$ defined by minimizing values of ω_0 and ω_1 for each (t,x,V) are lost.

5 Monotonicity conditions for nonsmooth Lyapunov type functions

Now let $\varphi(t,x,V)$ be a continuous function. Then monotonicity conditions are formulated via Hamilton–Jacobi type proximal inequalities. Let us consider only three monotonicity properties that are often used for optimal impulsive control problems.

Denote by $\partial_P\varphi(t,x,V)$ and $\partial^P\varphi(t,x,V)$ the proximal subdifferential and superdifferential of the function φ at the point (t,x,V); moreover, denote by $N_Q^P(t,x,V)$ the proximal normal cone to Q_φ at (t,x,V).

Let us recall [Clarke, Ledyaev, Stern, and Wolenski, 1998; Vinter, 2000] that a vector $p \in \mathbb{R}^k$ is called a proximal subgradient of $y \to \varphi(y)$ at a point y if there exist a neighborhood Q of the point y and a constant $c > 0$ such that

$$\varphi(z) \geq \varphi(y) + \langle p, z-y \rangle - c|z-y|^2 \quad \forall\, z \in Q.$$

This inequality implies that locally (in a neighbourhood of y) φ has a quadratic lower support function at the point y with gradient p at this point. The proximal subdifferential $\partial_P \varphi(y)$ consists of all such subgradients. It may be an empty set; in this case, the respective proximal inequalities given below are assumed to hold automatically at the point y. Note that $\partial_P \varphi(y) \subset \{\nabla \varphi(y)\}$ for a differentiable φ; moreover, if φ is twice continuously differentiable at the point y, then this inclusion turns into an equality. The proximal superdifferential $\partial^P \varphi$ is introduced in an antisymmetric way and is formally defined by the equality $\partial^P \varphi(y) = -\partial_P\big(-\varphi(y)\big)$.

Now let us consider the following three systems of conditions, which consist of proximal inequalities of a Hamilton–Jacobi type.

Condition (A):

(A1) $p_t + h_0(t,x,p_x) \geq 0, \quad p_V + h_1(t,x,p_x) \geq 0$
$\forall\, p = (p_t, p_x, p_V) \in \partial_P \varphi(t,x,V)$,
$\forall\, (t,x,V) \in (a,b) \times \mathbb{R}^n \times (0,+\infty)$;

(A2) $p_t + h_0(t,x,p_x) \geq 0$
$\forall\, (p_t, p_x) \in \partial_P \varphi^{V_0}(t,x)$,
$\forall\, (t,x) \in (a,b) \times \mathbb{R}^n$;

(A3) $p_V + h_1(t,x,p_x) \geq 0$
$\forall\, (p_x, p_V) \in \partial_P \varphi^t(x,V)$,
$\forall\, (t,x,V) \in \{a;b\} \times \mathbb{R}^n \times (0,+\infty)$.

Condition (B):

(B1) $\max\limits_{\substack{\omega_0,\omega_1 \geq 0\\ \omega_0+\omega_1=1}} \big[\big(p_t + \mathcal{H}_0(t,x,p_x)\big)\omega_0 + $
$+\big(-p_V + \mathcal{H}_1(t,x,p_x)\big)\omega_1\big] \geq 0$
$\forall\, p = (p_t, p_x, p_V) \in \partial^P \varphi(t,x,V)$,
$\forall\, (t,x,V) \in (a,b) \times \mathbb{R}^n \times (0,+\infty)$;

(B2) $p_t + \mathcal{H}_0(t,x,p_x) \geq 0$
$\forall\, (p_t, p_x) \in \partial^P \varphi^{V_0}(t,x)$,
$\forall\, (t,x) \in (a,b) \times \mathbb{R}^n$;

(B3) $-\,p_V + \mathcal{H}_1(t,x,p_x) \geq 0$
$\forall\, (p_x, p_V) \in \partial^P \varphi^t(x,V)$,
$\forall\, (t,x,V) \in \{a;b\} \times \mathbb{R}^n \times (0,+\infty)$.

Condition (C):

(C1) $\min\limits_{\substack{\omega_0,\omega_1 \geq 0\\ \omega_0+\omega_1=1}} \big[\big(p_t + h_0(t,x,p_x)\big)\omega_0 + $
$+\big(p_V + h_1(t,x,p_x)\big)\omega_1\big] \leq 0$
$\forall\, p = (p_t, p_x, p_V) \in \partial^P \varphi(t,x,V)$,
$\forall\, (t,x,V) \in (a,b) \times \mathbb{R}^n \times (0,+\infty)$;

(C2) $p_t + h_0(t,x,p_x) \leq 0$
$\forall\, (p_t, p_x) \in \partial^P \varphi^{V_0}(t,x)$,
$\forall\, (t,x) \in (a,b) \times \mathbb{R}^n$;

(C3) $p_V + h_1(t,x,p_x) \leq 0$
$\forall\, (p_x, p_V) \in \partial^P \varphi^t(x,V)$,
$\forall\, (t,x,V) \in \{a;b\} \times \mathbb{R}^n \times (0,+\infty)$.

Theorem 1.

1) φ *is strongly increasing iff condition* (A) *holds;*
2) φ *is weakly increasing iff condition* (B) *holds;*
3) φ *is weakly predecreasing iff condition* (C) *holds.*

6 Conclusion

In this paper inequalities of Hamilton–Jacobi type for nonlinear impulsive control systems with trajectories of bounded variation has been considered. Hamilton–Jacobi inequalities and equations play a fundamental role in the control theory, namely in the problems of stability, invariance, and attainability of controlled dynamical systems (see, for example, [Aubin and Cellina, 1984; Bardi and Capuzzo-Dolcetta, 1997; Clarke, Ledyaev, Stern, and Wolenski, 1998; Gurman, 1997; Guseinov and Ushakov, 1990; Krotov, 1996; Krotov and Gurman, 1973; Lyapunov, 1892, 1992; Subbotin, 1995; Vinter, 2000; Motta and Rampazzo, 1996; Matos, Pereira and Silva; Pereira and Silva, 2002; Pereira, Silva and Oliveira, 2008]). Such applications essentially use the properties of the weak and strong monotonicity of solutions of Hamilton–Jacobi inequalities. These solutions are interpreted as Lyapunov type functions. Some applications of the Lyapunov type functions for impulsive optimal control problems are in [Dykhta and Samsonyuk, 2010; Samsonyuk, 2010; Dykhta and Samsonyuk, 2011; Samsonyuk, 2011], where some families of the strongly and weakly monotone Lyapunov type functions are applied to necessary and sufficient global optimality conditions corresponding to the approach of the canonical Hamilton–Jacobi theory.

Acknowledgements

The research is supported by the Russian Foundation for Basic Research project 14-01-00699.

References

Arutyunov, A.V., Karamzin, D. Yu., Pereira, F.L. (2010). On constrained impulsive control problems. *J. Math. Sci.*, **165**, pp. 654–688.

Aubin, J-P, Cellina, A. (1984). *Differential inclusions*. Springer-Verlag, Berlin.

Bardi, M, Capuzzo-Dolcetta, I. (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhauser, Boston.

Bressan, A., Rampazzo, F. (1988). On differential systems with vector-valued impulsive controls. *Boll. Un. Mat. Ital.*, B(7), **2**, pp. 641–656.

Clarke, F., Ledyaev, Yu., Stern, R., and Wolenski, P. (1998). *Nonsmooth analysis and control theory*. In Graduate Texts in Mathematics, vol. 178. Springer-Verlag, New York.

Dykhta, V.A., Samsonyuk, O.N. (2010). Hamilton–Jacobi inequalities in control problems for impulsive dynamical systems. In: *Proceedings of the Steklov Institute of Mathematics*, **271**, pp. 86–102.

Dykhta, V.A., Samsonyuk, O.N. (2011). Some applications of Hamilton-Jacobi inequalities for classical and impulsive optimal control problems. *European Journal of Control,* **17**, pp. 1–15.

Gurman, V.I. (1997). *Extension principle in optimal control problems*. Fizmatlit, Moscow.

Guseinov, Kh.G, Ushakov, V.N. (1990). Strongly and weakly invariant sets with respect to differential inclusion, their derivatives, and application to control problems. *J Differential Equations*, **26**, pp. 1399–1405.

Krotov, V.F. (1996). *Global methods in optimal control theory*. Monographs and Textbooks in Pure and Applied Mathematics, **195**. Marcel Dekker, New York.

Krotov, V.F, Gurman, V.I. (1973). *Methods and problems of optimal control*. Nauka, Moscow [in Russian].

Lyapunov, A. ([1892], 1992). *The general problem of the stability of motion*. Taylor and Francis, London.

Matos, A.C., Pereira, F.L., and Silva, G.N. (2002). *Hamilton-Jacobi conditions for an impulsive control problem*. In: Nonlinear Control Systems, Fevereiro. pp. 1297–1302.

Miller, B.M., Rubinovich, E.Ya. (2003). *Impulsive controls in continuous and discrete-continuous systems*. Kluwer Academic Publishers, New York.

Miller, B.M. (1993) Method of discontinuous time change in problems of control of impulse and discrete-continuous systems. *Autom. Remote Control*, **54**, pp. 1727–1750.

Miller, B.M. (1996). The generalized solutions of nonlinear optimization problems with impulse control. *SIAM J. Control Optim.*, **34**, pp. 1420–1440.

Motta, M., Rampazzo, F. (1995). Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential Integral Equations*, **8**, pp. 269–288.

Motta, M., Rampazzo, F. (1996). Dynamic programming for nonlinear systems driven by ordinary and impulsive control. *SIAM J. Control Optim.*, **34**, pp. 199–225.

Pereira, F.L., Silva, G.N. (2000). Necessary conditions of optimality for vector-valued impulsive control problems. *Systems and Control Lett*, **40**, pp. 205–215.

Pereira, F.L., Silva, G.N. (2002). Stability for impulsive control systems. *Dyn. Syst.*, **17**, pp. 421–434.

Pereira. F.L., Silva, G.N., and Oliveira, V. (2008). Invariance for impulsive control systems. *Autom. Remote Control*, **69**, pp. 788–800.

Samsonyuk, O.N. (2010). Compound Lyapunov type functions in control problems of impulsive dynamical systems. In: *Trudy Inst Mat Mekh, UrO RAN, Ekaterinburg*, **16**, pp. 170–178.

Samsonyuk, O. (2011). Lyapunov type functions for nonlinear impulsive control systems: monotonicity conditions and applications. *Book of Abstracts of the 5th International Conference on Physics and Control*, pp. 87.

Samsonyuk, O.N. (2014). Monotonicity of Lyapunov type functions for impulsive control systems. *The bulletin of Irkutsk state university. Series "Mathematics"*, **7**, pp. 104–123.

Samsonyuk, O.N. (2015). Invariant sets for the nonlinear impulsive control systems. *Autom. Remote Control*, **76**, pp. 405–418.

Silva, G.N., Vinter, R.B. (1996). Measure differential inclusions. *J. of Mathematical Analysis and Applications*, **202**, pp. 727–746.

Subbotin, A.I. (1995). *Generalized solutions of first-order PDEs: the dynamical optimization perspective*. Birkhauser, Boston.

Vinter, R.B. (2000). *Optimal control*. Birkhauser, Boston-Basel-Berlin.

Zavalishchin, S.T., Sesekin, A.N. (1991). *Impulse processes: models and applications*. Nauka, Moscow.

Zavalishchin, S.T., Sesekin, A.N. (1997). *Dynamic impulse systems: theory and applications*. Kluwer Academic Publishers, Dordrecht.